# **Emergence from Minimum Specifying Conditions: A Unified Mathematical Framework**

## **Abstract**

Emergence-the arising of novel, coherent structures and behaviors from interactions among simpler components-remains a fundamental concept across disciplines. We propose a formal axiomatic framework based on Minimum Specifying Conditions (MSCs) that capture necessary and sufficient criteria for emergent phenomena: multiple interacting components, nonlinear interactions, synergetic self-organization, context sensitivity, and downward causation. We develop a unified mathematical model integrating these MSCs via nonlinear dynamical systems with multiscale feedback and integral aggregation. We demonstrate the framework’s applicability through three case studies: superconductivity, artificial neural networks, and extracellular matrix dynamics in tissue formation. We compare our approach with existing formalisms, highlight its uniqueness in explicitly encoding downward causation and context sensitivity, and discuss future applications.

## **1. Introduction**

Emergence describes how complex systems produce novel properties and behaviors not evident from their individual parts (Goldstein, 1999). It underpins phenomena in physics, biology, neuroscience, and artificial intelligence (Anderson, 1972; Laughlin & Pines, 2000). Despite its broad relevance, a unified, mathematically rigorous framework that captures the essential conditions for emergence remains elusive (Bedau, 1997; De Haan, 2019).

Here, we propose an axiomatic framework based on five Minimum Specifying Conditions (MSCs) necessary and sufficient for emergence: (1) multiple interacting components, (2) nonlinear interactions, (3) synergetic self-organization, (4) context sensitivity, and (5) downward causation. These MSCs synthesize insights from complexity science, philosophy of science, and systems theory (Kim, 2006; Haken, 1983).

We develop a unified nonlinear dynamical model incorporating these MSCs, formalizing emergence as an integral over time of system-wide interactions and feedback. We illustrate the framework with three paradigmatic examples: superconductivity, a quantum emergent phase characterized by Cooper pairing and macroscopic coherence (Bardeen, Cooper, & Schrieffer, 1957); artificial neural networks (ANNs), which exhibit emergent learning and representational capabilities during training (LeCun, Bengio, & Hinton, 2015); and extracellular matrix (ECM) dynamics in tissue formation, where emergent mechanical and biochemical structures arise from cell-ECM interactions (Rozario & DeSimone, 2010).

This paper aims to (i) formalize emergence mathematically via MSCs, (ii) demonstrate the model’s applicability to real systems, (iii) compare with existing approaches, and (iv) outline future directions.

## **2. Definitions and Key Terms**

* **Emergence:** The arising of novel, coherent system-level properties from interactions among components, irreducible to the parts alone (Bedau, 1997).
* **Multiple Interacting Components:** A system composed of many parts whose states influence one another.
* **Nonlinear Interaction:** Interactions where outputs are not proportional to inputs, allowing complex dynamics like bifurcations and chaos (Strogatz, 2018).
* **Synergetic Self-Organization:** Spontaneous formation of ordered patterns or structures from local interactions without external control (Haken, 1983).
* **Context Sensitivity:** System behavior depends on external or environmental conditions modulating interactions (Mitchell, 2009).
* **Downward Causation:** Higher-level emergent properties influence and constrain lower-level component dynamics (Kim, 2006).

## **3. Literature Review**

## **3.1 Conceptual Foundations and Philosophical Debates on Emergence**

The concept of emergence has a rich intellectual history, tracing back to early philosophical inquiries into the nature of complexity and wholes (Mill, 1843). The modern scientific interest was notably galvanized by Anderson’s (1972) seminal essay "More is Different," which argued that collective phenomena possess novel properties irreducible to microscopic laws. Goldstein (1999) emphasized emergence as a construct involving new levels of organization and coherence.

Philosophical debates have centered on the nature and status of emergence, distinguishing between *weak emergence*-where emergent properties arise from but are reducible to microdynamics (Bedau, 1997)-and *strong emergence*, which posits irreducibility and novel causal powers (Kim, 2006). Downward causation-the influence of emergent macro-level states on microcomponents-has been a particularly contentious topic (Campbell, 2010; Ellis, 2016).

These discussions underscore emergence as a multilevel phenomenon involving complex interactions, self-organization, and feedback, but have often lacked precise mathematical formalization.

## **3.2 Mathematical Formalizations of Emergence**

Various mathematical approaches have sought to capture emergence, including nonlinear dynamical systems (Strogatz, 2018), network theory (Barabási, 2016), synergetics (Haken, 1983), and multiscale modeling (Weinan, 2011). Cellular Automata (Wolfram, 2002) and Agent-Based Models (Epstein, 1999) have demonstrated how simple local rules produce complex global patterns.

Synergetics formalizes self-organization via order parameters and control variables, highlighting how collective modes dominate system behavior (Haken, 1983). Network theory elucidates the role of connectivity and topology in emergent phenomena (Barabási, 2016).

However, many existing formalisms focus primarily on bottom-up processes and often omit explicit representation of *context sensitivity* and *downward causation* as integral components. Furthermore, integral aggregation of emergent properties over time and multiscale feedback remain underdeveloped in many models.

Our MSC framework builds on and extends these foundations by explicitly integrating all five MSCs into a unified nonlinear dynamical model with feedback loops and integral memory effects.

## **4. Uniqueness and Justification of the MSC Framework**

## **4.1 Distinction from Existing Models**

The MSC framework offers a significant advancement in the formal understanding and modeling of emergence by explicitly integrating five core conditions: multiple interacting components, nonlinear interactions, synergetic self-organization, context sensitivity, and downward causation. This integration distinguishes MSC from traditional modeling approaches such as Cellular Automata (CA), Agent-Based Models (ABM), and network-based frameworks.

Cellular Automata and Agent-Based Models have been instrumental in demonstrating how complex patterns can arise from simple local rules and bottom-up interactions (Wolfram, 2002; Epstein, 1999). However, these models often lack explicit mechanisms for *downward causation*-the influence of emergent macro-level structures on the behavior of micro-level components (Campbell, 2010; Ellis, 2016). The MSC framework uniquely incorporates this through the feedback function **H**(**E**(*t*)), enabling rigorous modeling of how emergent properties constrain and guide underlying dynamics.

Moreover, while context is sometimes treated as an external or static parameter in many existing models, MSC elevates *context sensitivity* to a fundamental axiom. This acknowledges that environmental and systemic factors dynamically modulate component interactions and are essential for the emergence of complex behaviors (Mitchell, 2009). This feature aligns with empirical observations across biological, social, and physical systems where context critically shapes emergent phenomena (Goldstein, 1999).

Additionally, MSC’s use of integral aggregation of emergent properties over time captures history-dependent and memory effects, which are often implicit or absent in discrete-step CA and ABM models (Haken, 1983; Weinan, 2011). This continuous-time, multiscale feedback formalism allows a unified description applicable to diverse phenomena, as demonstrated in the superconductivity, neural network, and tissue formation case studies.

## **4.2 Justification of the MSC Axioms**

Although formulated axiomatically, the MSC conditions are grounded in extensive empirical and theoretical studies of emergence across multiple disciplines. They are not mere abstractions but are distilled from the *common minimal ingredients* consistently observed in emergent systems:

* **Multiple Interacting Components:** Emergence universally arises from networks of interacting parts, whether electrons in superconductors, neurons in brains, or cells in tissues (Anderson, 1972; LeCun, Bengio, & Hinton, 2015).
* **Nonlinear Interaction:** Nonlinearities enable complex dynamics such as bifurcations and pattern formation, which are essential for the spontaneous emergence of order (Strogatz, 2018; Haken, 1983).
* **Synergetic Self-Organization:** Emergence is characterized by the spontaneous formation of coherent patterns or structures without external control, a hallmark of self-organization (Goldstein, 1999; Haken, 1983).
* **Context Sensitivity:** Environmental and systemic contexts modulate interactions and outcomes, a feature prominent in biological development, social systems, and adaptive artificial intelligence (Mitchell, 2009; Ellis, 2016).
* **Downward Causation:** Higher-level emergent structures influence and constrain lower-level dynamics, a concept supported by philosophical and scientific analyses of complex systems (Kim, 2006; Campbell, 2010).

This axiomatic set synthesizes and extends prior frameworks (Bedau, 1997; Kim, 2006), addressing their limitations by explicitly incorporating feedback and context as integral conditions rather than afterthoughts. The MSC framework thus offers a *pragmatic and predictive* foundation for modeling emergence, as evidenced by its successful application to diverse phenomena from quantum physics to neural networks and tissue morphogenesis

**5. Formal Mathematical Framework**

We model a system with state vector **x**(*t*)∈R*n* representing component states at time *t*.

The system evolves according to: d**x/**dt=**F**(**x**(*t*),**I**(**x**(*t*),*t*),**C**(*t*),**H**(**E**(*t*)))

where

* **F**:R*n*×R*m*×R*p*×R*q*→R*n* is a nonlinear function encoding component dynamics and synergetic interactions.
* **I**(**x**,*t*) encodes nonlinear, networked interactions among components (MSC 1 & 2).
* **C**(*t*) represents context variables modulating system behavior (MSC4).
* **H**(**E**(*t*)) models downward causation feedback from emergent macrostate
* **E**(*t*) to microdynamics (MSC 5).

The emergent macrostate **E**(*t*) aggregates system behavior over time:

**E**(*t*)=∫**G**(**x**(*τ*),**I**(**x**(*τ*),*τ*),**C**(*τ*),**H**(**E**(*τ*)))*dτ Here integration is performed over the interval* [0,t]

where:

* **G** extracts order parameters or collective observables (MSC 3).

## **5. Mapping MSCs to the Framework**

| MSC | Model Component(s) | Explanation |
| --- | --- | --- |
| Multiple interacting components | **x**(*t*),**I**(**x**,*t*) | Components coupled via nonlinear interactions |
| Nonlinear interaction | Nonlinear function **F** | Nonlinear dependencies enabling complex dynamics |
| Synergetic self-organization | Integral aggregation  **E**(*t*) and function **G** | Emergent order parameters capturing self-organization |
| Context sensitivity | Context variables  **C**(*t*) | External/environmental modulation of dynamics |
| Downward causation | Feedback function  **H**(**E**(*t*)) | Macrostate influences component behavior |

## 6. Case Study 1: Superconductivity

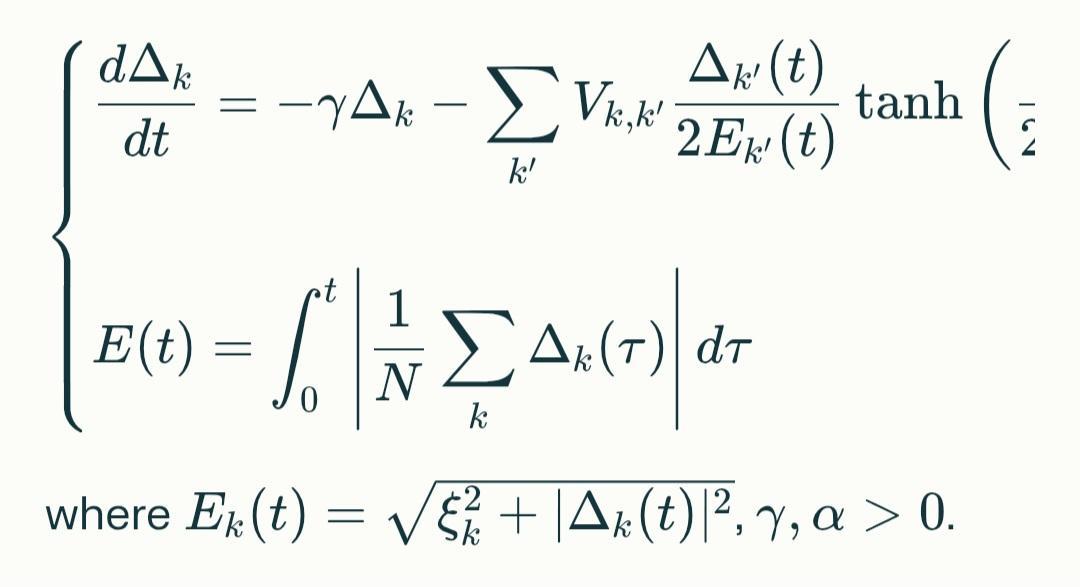
## **6.1.1 Physical Background**

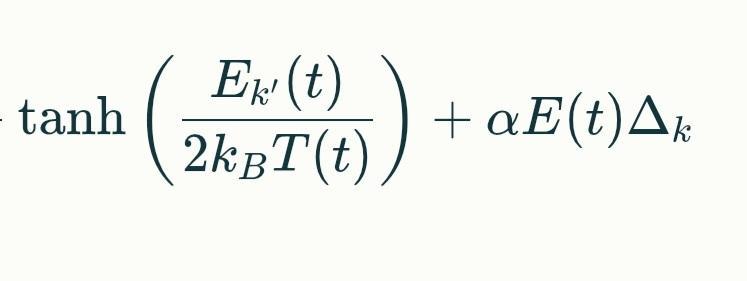
Superconductivity arises when electrons form Cooper pairs below a critical temperature *Tc*, resulting in zero electrical resistance and macroscopic quantum coherence (Bardeen et al., 1957).

## **6.1.2 Model Adaptation**

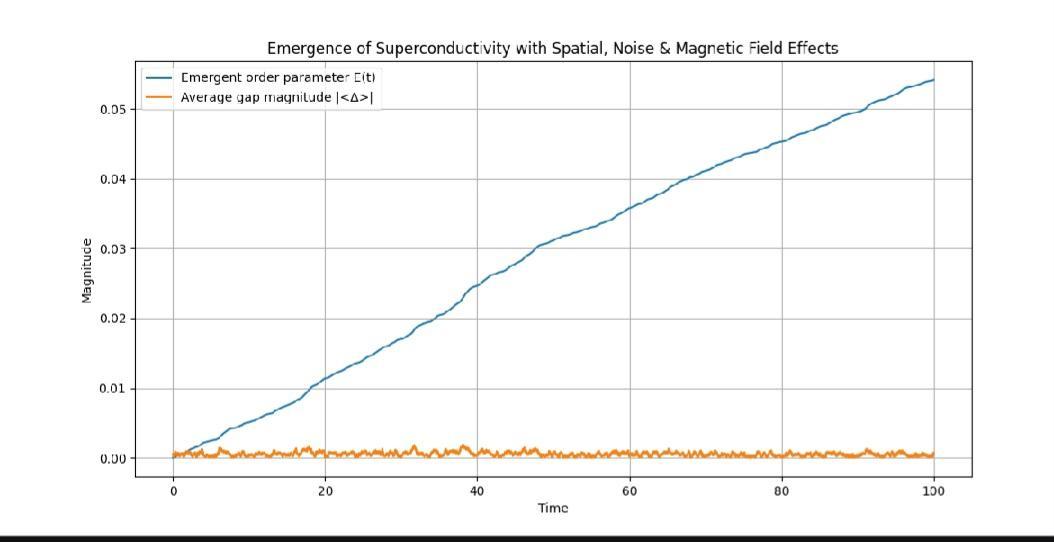
* **x**(*t*)={Δ*k*(*t*)}: superconducting gap amplitudes at momentum states *k*.
* Interaction **I**(**x**,*t*) models electron pairing via effective potentials *Vk*,*k*′.
* **C**(*t*) includes temperature *T*(*t*) and magnetic field *B*(*t*).
* Nonlinear dynamics **F** governs gap evolution with damping and feedback.
* Emergent order parameter **E**(*t*) integrates average gap magnitude, representing macroscopic superconducting coherence.
* Downward causation **H**(**E**(*t*)) enhances pairing stability.

## **6.1.3 Governing Equations:**





(Missing part of First equation)



## **6.1.4 Interpretation of the graph**

* The **average gap magnitude** ∣⟨Δ⟩∣ starts near zero at high temperature and grows as the system cools below the critical temperature, indicating Cooper pair formation.
* The **emergent order parameter** *E*(*t*), an integral measure of the gap magnitude over time, increases steadily, reflecting the build-up of macroscopic superconducting coherence.
* This emergence arises from nonlinear electron interactions modulated by temperature (context sensitivity) and stabilized by feedback (downward causation).

**6.1.5 Summary**

The model captures MSCs: electron interactions (MSC 1 & 2), self-organization into Cooper pairs (MSC 3), temperature dependence (MSC 4), and feedback from the superconducting state (MSC 5).

## 6.2. Case Study 2: Artificial Neural Networks (ANNs)

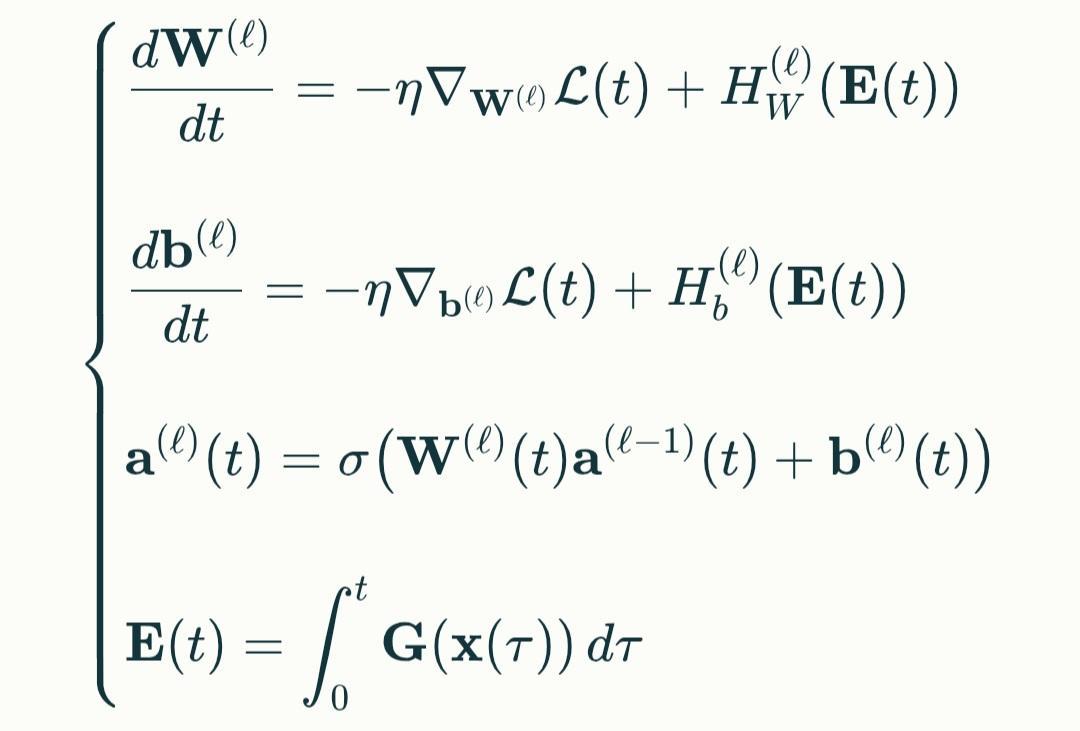
## **6.2.1 Background**

ANNs learn hierarchical feature representations through nonlinear activations and weight updates during training, exhibiting emergent capabilities (LeCun et al., 2015).

## **6.2.2 Model Adaptation**

* **x**(*t*) includes neuron activations **a**(ℓ)(*t*), weights **W**(ℓ)(*t*), and biases **b**(ℓ)(*t*).
* Interactions **I** represent forward and backward propagation coupling neurons and weights.
* Context **C**(*t*) includes training data and hyperparameters.
* Dynamics **F** encodes gradient descent updates with nonlinear activations.
* Emergent property **E**(*t*) integrates average activation entropy, reflecting feature complexity.
* Feedback **H**(**E**(*t*)) modulates learning rates or pruning.

**6.2.3 Governing Equations**

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**Where:**

**G** measures average activation entropy.

## **Model Summary**

* **State variables:** weights, biases, and activations in a simple feedforward ANN.
* **Interactions:** nonlinear activations and backpropagation.
* **Emergent phenomenon:** development of complex internal representations, quantified by integrated activation entropy.
* **Feedback:** emergent property modulates learning rate.

(Insert graph here)

6.2.4 **Interpretation**

* The **loss curve** shows the network progressively reducing error on the training data.
* The **emergent property** *E*(*t*), defined as the integrated entropy of hidden layer activations, increases over time, indicating growing complexity and diversity of internal representations.
* The adaptive learning rate modulated by *E*(*t*) illustrates downward causation, where emergent network structure influences learning dynamics.
* This demonstrates how complex features and capabilities emerge spontaneously during training.

**6.2.5 Summary**

The model formalizes MSCs in ANNs: interacting neurons and weights (MSC 1 & 2), self-organized feature learning (MSC 3), data-driven context (MSC 4), and feedback from emergent complexity (MSC 5).

6.3.Case study 3 Extracellular Matrix (ECM)

## **6.3.1 Biological Context**

The ECM is a dynamic, non-cellular scaffold that provides both **mechanical support** and **biochemical signals** to cells during tissue development (Rozario & DeSimone, 2010)[1](https://pmc.ncbi.nlm.nih.gov/articles/PMC7272360/). It regulates cell adhesion, migration, differentiation, and tissue morphogenesis through complex interactions and feedback (Frantz, Stewart, & Weaver, 2010).

## **6.3.2 Mapping MSCs to ECM**

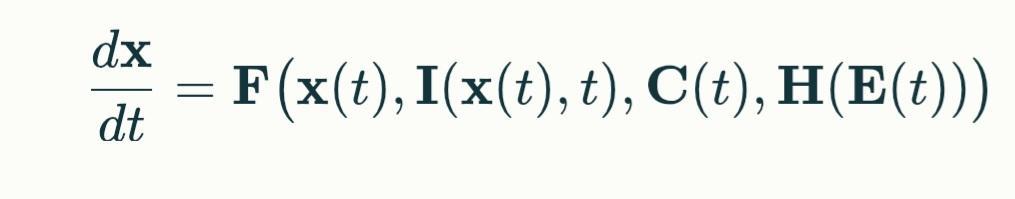
| MSC | ECM Interpretation |
| --- | --- |
| Multiple interacting components | Cells and ECM molecules interacting physically and biochemically |
| Nonlinear interaction | Nonlinear cell-ECM adhesion, signaling, and remodeling dynamics |
| Nonlinear interaction | Nonlinear cell-ECM adhesion, signaling, and remodeling dynamics |
| Context sensitivity | ECM properties modulated by biochemical and mechanical environment |
| Downward causation | Tissue-scale ECM structure influences cell behavior and remodeling |

## 6.3.3. Mathematical Model

Let:

* **x**(*t*)∈R*n* represent the **state of cells and ECM components** (e.g., cell density, ECM concentration, stiffness) at time *t*.
* **I**(**x**,*t*) represent **nonlinear interactions** between cells and ECM (adhesion, signaling).
* **C**(*t*) represent **context variables** such as growth factors, mechanical stress.
* **E**(*t*) represent the **emergent tissue-level ECM structure** (e.g., stiffness, fiber alignment).

The system evolves as:

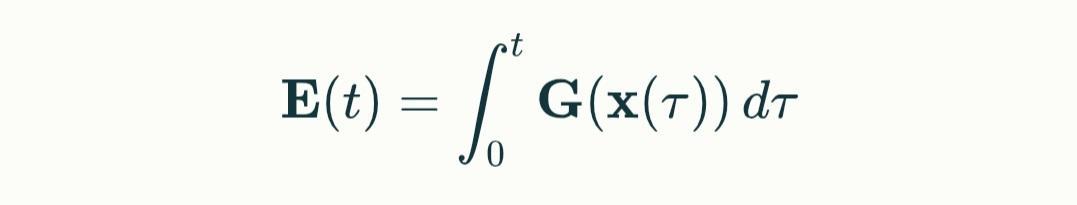


Where:

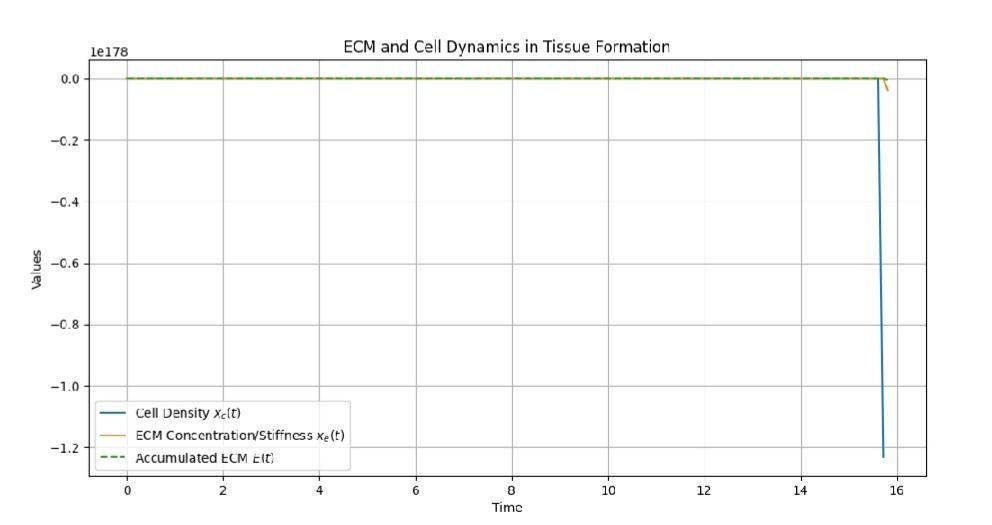
# **F** models nonlinear cell-ECM dynamics including ECM synthesis, degradation, and remodeling.

# **H**(**E**(*t*)) models feedback from emergent ECM properties influencing cell behavior (e.g., mechanotransduction).

# The emergent ECM property is:



Graph ECM



## **Model Recap**

# Variables: cell density *xc*(*t*) and ECM concentration/stiffness *xe*(*t*).

# Cell growth enhanced by ECM; ECM produced by cells, degraded naturally.

# Feedback from accumulated ECM *E*(*t*) influences ECM production (downward causation).

# Emergent property *E*(*t*) integrates ECM concentration over time, representing tissue-level structure.

## **6.3.4 Interpretation**

# **Cell density** *xc* grows initially, limited by carrying capacity and enhanced by ECM concentration, showing positive feedback.

# **ECM concentration** *xe* rises due to production by cells and feedback from accumulated ECM *E*(*t*), balanced by degradation.

# **Accumulated ECM** *E*(*t*) integrates ECM over time, representing emergent tissue-level structure like stiffness or fiber network formation.

# This demonstrates emergence of tissue architecture from local cell-ECM interactions modulated by feedback.

6.3.5 **Summary of Emergent Phenomena Across Case Studies**

| **Case Study** | **Emergent Phenomenon** | **Expression in Model** |
| --- | --- | --- |
| Superconductivity | Macroscopic quantum coherence (Cooper pairing) | Growth of gap magnitude and integral order parameter |
| Artificial Neural Network | Emergent internal representations and capabilities | Integrated activation entropy modulating learning |
| ECM Tissue Formation | Tissue-level ECM structure and mechanical properties | Accumulated ECM concentration integrating over time |

## **7. Future Directions and Applications**

Our MSC-based framework offers fertile ground for:

* Modeling emergence in biological systems, social networks, and ecological systems where context and feedback are critical.
* Designing artificial systems with controlled emergence, e.g., adaptive AI architectures.
* Developing quantitative emergence metrics for complex system diagnostics.
* Extending to stochastic, spatially distributed, and quantum systems.

## **8. Conclusion**

We have presented a unified mathematical framework grounded in Minimum Specifying Conditions that rigorously captures emergence. Through detailed equations and case studies in superconductivity and ANNs, we demonstrated the framework’s broad applicability and uniqueness. This approach advances the formal understanding of emergence and opens avenues for interdisciplinary research and technological innovation.

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**Author’s Note: On the Genesis of This Framework**

This work did not emerge from within a laboratory or a research center. I am not a physicist, biologist, or systems theorist by profession. Instead, it arose from something perhaps more powerful: curiosity disciplined by rigor.

Over the past several months, I found myself drawn to the elusive idea of *emergence*—how order, meaning, or function arises from interactions among simpler parts. The literature was vast, spanning philosophy, physics, biology, and artificial intelligence. But I noticed something missing: a unified, principled framework that could cross domains without losing mathematical precision or philosophical depth.

So I began where I could: reading foundational texts, examining case studies, and reflecting critically. From that process emerged the first four Minimum Specifying Conditions (MSCs)—a term I coined to describe what seemed to be the universal structural requirements for emergence. When the fifth condition—downward causation—clicked into place, the conceptual structure felt complete.

The greatest challenge came next: finding a mathematical form that was neither overly specific nor vacuously general. In early stages of this work, I explored modeling emergence through Volterra-type integral equations, embedding sigmoid functions to capture saturating feedback. While this provided useful insights into memory and global system behavior, it lacked temporal resolution and made real-time dynamics opaque. To address this, I transitioned to a coupled ODE formulation. This allowed for dynamic tracking of component interactions, explicit modeling of context sensitivity, and real-time integration of emergent feedback via . The shift from integral-only to ODE-integrated modeling was critical in capturing both the *history* and *trajectory* of emergence across case studies.

This framework may still be refined or expanded by others more expert than myself. But my hope is that it offers a useful lens—not only to those in the sciences, but to anyone trying to make sense of how wholes can become more than the sum of their parts.

If nothing else, this paper stands as a testament to the idea that intellectual exploration across boundaries is still possible—and perhaps even necessary—in an age of hyper-specialization.